# Cycle Problems and Path Problems in Digraphs

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# Outline

# 01 Definitions and Notations

# Basic Definitions 1

- **Graph $G$: an ordered pair $(V(G), E(G))$, in which $V(G)$ is the set of vertices and $E(G)$ is the set of edges, which consists of unordered pairs of vertices.**
- **Digraph $D$: $(V(D), A(D))$, $V(D)$ set of vertices, $A(D)$ set of arcs, which consists of ordered pairs of vertices.**

# Basic Definition 2

- **Path.**
- **Cycle.**
- **Length of a path or cycle:** number of edges.
- **Hamiltonian path (cycle):** a path (cycle) that contains every vertex of the graph (digraph).
- **Traceable (Hamiltonian):** a graph (digraph) with a Hamiltonian path (cycle).

# Definition & Notations 1

- **Neighbor:** **two distinct vertices that are adjacent by an edge (arc) are neighbors.**
- **Neighborhood $N(v)$ & Degree $d(v)$:** **the set of neighbors of a vertex $v$, & $d(v) = |N(v)|$.**
- **In- & out-neighbor.**
- **In- & out-neighborhood.** $N^-(v)$, $N^+(v)$.
- **In-, out-degree & (total) degree.** $d^-(v) = |N^-(v)|$, $d^+(v) = |N^+(v)|$, $d(v) = d^-(v) + d^+(v)$.

# Definition & Notations 2

- $\delta(G)$, $\delta(D)$, $\delta^+(D)$ and $\delta^-(D)$: The minimum degree of the vertices in a graph $G$, minimum (total) degree in a digraph $D$, the minimum out-degree and in-degree of the vertices in $D$.
- $\delta^0(D)$: $\min\{\delta^+(D), \delta^-(D)\}$, minimum semi-degree.
- $\sigma_k(G)$: minimum degree sum of all combinations of $k$ independent vertices in a graph $G$. So $\delta(G) = \sigma_1(G)$.

# 02

# Cycle and Path Problems in General

# Classical Results (Density)

- **(Dirac, 1952)**

  **If $\delta(G) \geq n/2$, then $G$ is Hamiltonian.**

- **(Ore, 1960)**

  **If $\sigma_2(G) \geq n$, then $G$ is Hamiltonian.**

# Generalization – Weaker conditions

- **(Fan, 1984)**

  **If $G$ is 2-connected and for every distinct vertices $u$ and $v$ with $d(u, v) = 2$, $\max\{d(u), d(v)\} \geq n/2$, then $G$ is Hamiltonian.**

- **Connectivity:** **the minimum number of vertices to be removed from $G$ to make it disconnected.**
- **Distance $d(u, v)$:** **the length of the shortest path between $u$ and $v$ in $G$.**

# Generalization – Various conditions 1

- **$\sigma_k(G)$ for $k \geq 3$. $\sigma_k(G) \geq f(n)$.**
- **Neighborhood union.**
  $|N(u) \cup N(v)| = d(u) + d(v) - |N(u) \cap N(v)|$.
- **Size: number of edges. (example. $C(n\text{-}1, 2)+1$ for being Hamiltonian)**

# Generalization – Various conditions 2

- **Degree sequence.**
- **Forbidden (induced) subgraph.**
- **Closures.**
- **Toughness.**
- **Binding number.**

# Generalization - More Structures 1

- **Number of Hamiltonian cycles.**
- **Number of edge-disjoint Hamiltonian cycles.**
  - **Decomposition & packing.**
- **Powers of a Hamiltonian cycle.**
- **Long cycles (circumference).**
- **Dominating cycle.**
- **Anti-directed paths and anti-directed cycles.**
- **Factors.**

# Generalization - More Structures 2

- Pancyclicity.
- Vertex- (edge-, arc-)pancyclicity.
- Hamiltonian-connectedness.
- Panconnectedness.
- Cycle extendability.
- Path extendability.

# 03 Degree Conditions and Extremal Digraphs

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SPRINGER
MONOGRAPHS IN MATHEMATICS

# Digraphs

## Theory, Algorithms and Applications

Second Edition

Springer

# Degree Conditions 1

- **(Ghouila-Houri, 1960)**

  **If $\delta(D) \geq n$, then $D$ is Hamiltonian.**

- **(Corollary)**

  **If $\delta^0(D) \geq n/2$, then $D$ is Hamiltonian.**

- **(Woodall, 1972)**

  **If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices $x$ and $y$ such that there is no arc from $x$ to $y$, then $D$ is Hamiltonian.**

# Degree Conditions 2

- **(Meyniel, 1973)**

  **If $D$ is strong and $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices $x$ and $y$ in $D$, then $D$ is Hamiltonian.**

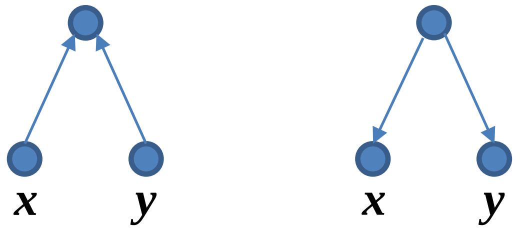
- **Strong: for every pair $x, y$ of distinct vertices there exist an $(x, y)$-path and a $(y, x)$-path.**

# Degree Conditions 3

- **(Bang-Jensen et al. 1996)**

  **If $D$ is strong and $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{x, y\}$. Then $D$ is Hamiltonian.**

- **Dominating, dominated pair.**



# Degree Conditions 4

- (Conjectured by Bang-Jensen et al. 1996)

  If $D$ is strong and $d(x) + d(y) \geq 2n - 1$ for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{x, y\}$. Then $D$ is Hamiltonian.

  If $D$ is strong and $d(x) + d(y) \geq 2n - 1$ for every pair of dominated non-adjacent vertices $\{x, y\}$. Then $D$ is Hamiltonian.

# Extremal Digraphs

- **(Ghouila-Houri, 1960)**

  **If $\delta(D) \geq n$, then $D$ is Hamiltonian.**

- **(Corollary)**

  **If $\delta^0(D) \geq n/2$, then $D$ is Hamiltonian.**

- **(Nash-Williams, 1969)**

  **Problem: describe all the extreme digraphs for Ghouila-Houri's theorem, i.e. the strong non-Hamiltonian digraphs of order $n$ and minimum degree $n$ - 1.**

**We do not say that the conclusion does not hold. We say that it holds with some exceptions!**

– Professor Akira Saito

# Extremal Digraphs for Ghouila-Houri's condition

- **(Thomassen, 1981)**

  **A structural result characterizing strong non-Hamiltonian digraphs of order $n$ and minimum degree $n$ - 1.**

- **(Darbinyan, 1986)**

  **If $D$ is a digraph of even order $n \geq 4$ such that $\delta(D) \geq n - 1$ and $\delta^0(D) \geq n/2 - 1$, then either $D$ is Hamiltonian or $D$ belongs to a non-empty finite family of exceptional digraphs.**

# Extremal Graphs for Woodall's condition

- **(Woodall, 1972)**

  **if $d^+(x) + d^-(y) \geq n$ for all pairs of vertices $x$ and $y$ such that there is no arc from $x$ to $y$, then $D$ is Hamiltonian.**

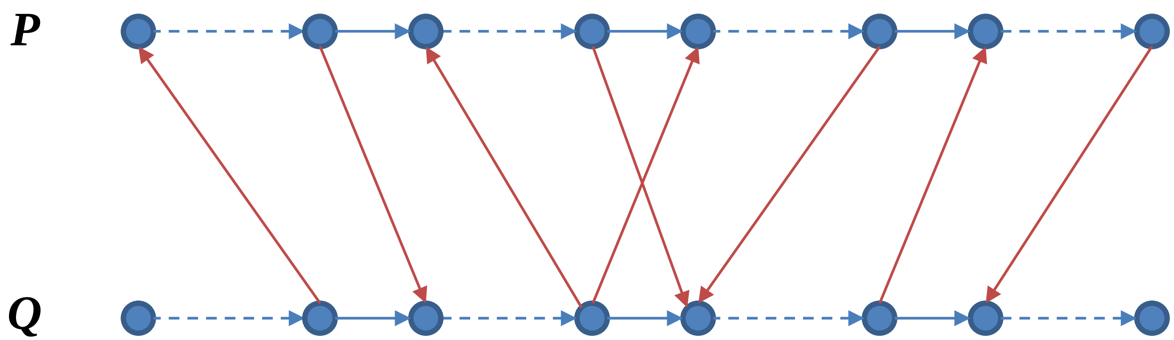
- **(Zhang, Zhang & Wen, SIDMA 2013)**

  **if $d^+(x) + d^-(y) \geq n - 1$ for all pairs of vertices $x$ and $y$ such that there is no arc from $x$ to $y$, then $D$ is Hamiltonian, unless $D$ belongs to one of four exceptional families of digraphs.**

# Techniques

- **Multi-insertion (Bang-Jensen et al., 1996)**

  **If a path $P$ can be multi-inserted in to a path (cycle, resp.) $Q$, then there exists a path $R$ with the same starting vertex and ending vertex of $Q$ (a cycle $R$, resp.), and $V(R)=V(P)\cup V(Q)$.**



# Further Problems

- **Give a complete characterization of the extremal digraphs of Ghoui-Houri's condition.**
- **Characterize the extremal digraphs of the conditions of Meyniel.**
- **Characterize the extremal digraphs of the conditions of Bang-Jensen et al.**

# 04 More Structures in Digraphs

# The "First Push"

- **(Bondy, 1971)**

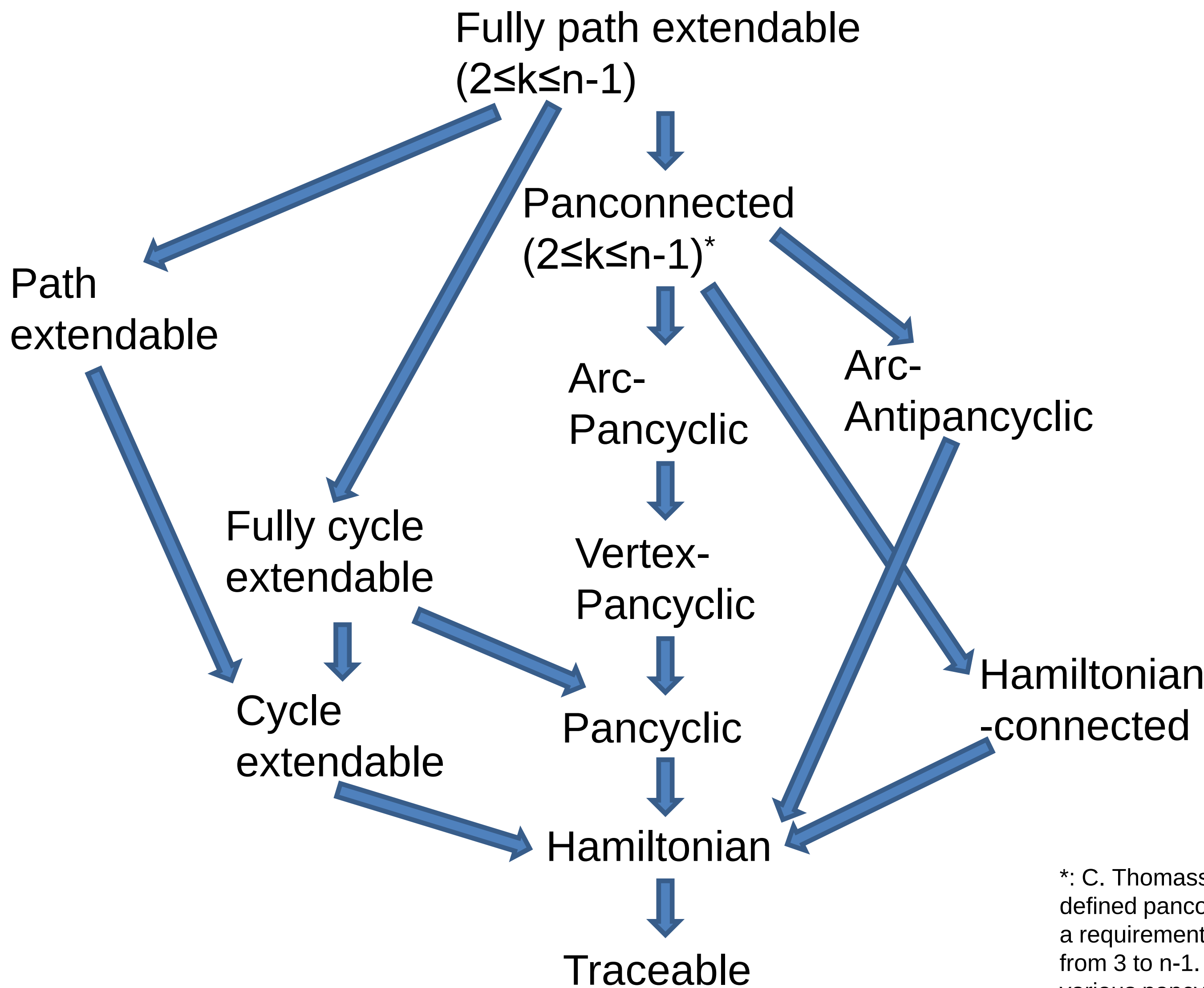
  **"Meta-conjecture": Almost any nontrivial Hamiltonian condition also implies pancyclicity.**

# Cycles

- **Pancyclic:** there exist cycles of every length from $3$ to $n$ in a graph (digraph).
- **Vertex- (edge-, arc-) pancyclic:** every vertex, (edge, arc) is contained in cycles of every length from $3$ to $n$.
- **Cycle extendable:** for every cycle $C$, there exists another cycle $C'$, such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$.
- **Fully cycle extendable**.

# Paths

- **Hamiltonian-connected:** between every two vertices there is a Hamiltonian path.
- **Panconnected:** between every two vertices there exist paths of every length from $3(2)$ to $n$ - $1$.
- **Path extendable:** for every path $P$, there exists another path $P'$ with the same starting and ending vertices, such that $V(P) \subset V(P')$ and $|V(P')| = |V(P)| + 1$.
- **Fully path extendable.**




*: C. Thomassen originally defined panconnecteness with a requirement of path of length from 3 to n-1. But to imply various pancyclicities we must have path from 2 to n-1 here.

# Pancyclicity: Degree Conditions

- **(Thomassen, 1977)**

  If $d(x) + d(y) \geq 2n$ whenever $x$ and $y$ are nonadjacent, then either $D$ has cycles of all lengths $2, 3, \ldots, n$ or $D$ belongs to some exception classes of digraphs.

- **(Alon & Gutin, 1997)**

  If $\delta^0(D) \geq n/2 + 1$, then $D$ is vertex-2-pancyclic. (every vertex is contained in cycles of every length from $2$ to $n$)

- **(Randerath et al., 2002)**

  If $\delta^0(D) \geq (n + 1)/2$, then $D$ is vertex-pancyclic.

# Pancyclicity: Size Conditions

- **(Häggkvist & Thomassen, 1976)**

  **Every Hamiltonian digraph on $n$ vertices and at least $n(n+1)/2-1$ arcs is pancyclic.**

# Cycle Extendability

- **Degree (Hendry, 1989)**

  **If $\delta(D) \geq 3n/2 - 2$, then $D$ is cycle extendable with some exceptions.**

  **If $\delta^+(D), \delta^-(D) \geq 2n/3 - 1$, then $D$ is cycle extendable with some exceptions.**

- **Size (Hendry, 1989)**

  **If $D$ has at least $(n - 1)^2$ arcs, then $D$ is cycle extendable. with some exceptions.**

  **If $D$ is strong and has more than $n^2 - 3n + 4$ arcs, then $D$ is cycle extendable.**

# Path Extendability

- **Implying relation (Zhang et al., SIDMA 2017)**

  **If $D$ is path extendable, then $D$ is cycle extendable.**

- **Degree (Zhang et al., SIDMA 2017)**

  **If $\delta(D) \geq 3n/2 - 1$, then $D$ is cycle extendable with some exceptions.**

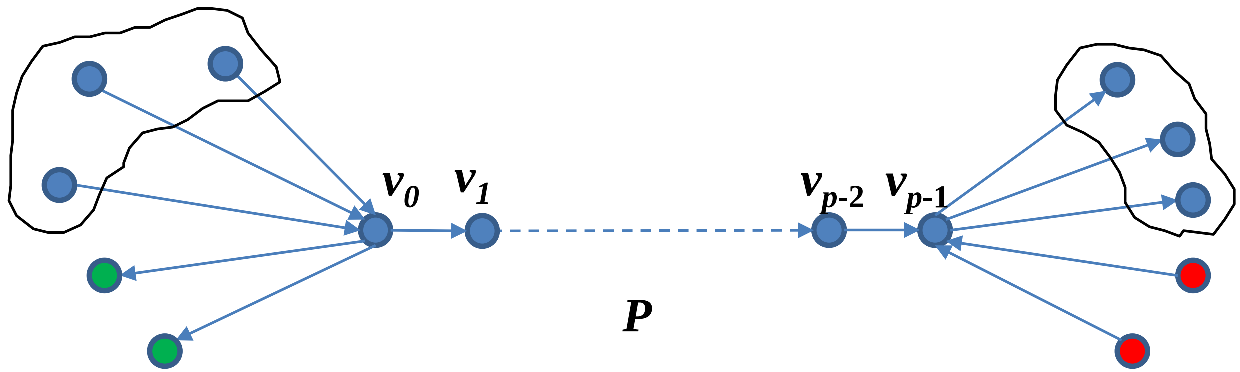
  **If $\delta^+(D), \delta^-(D) \geq (2n-2)/3$, then $D$ is cycle extendable with some exceptions.**

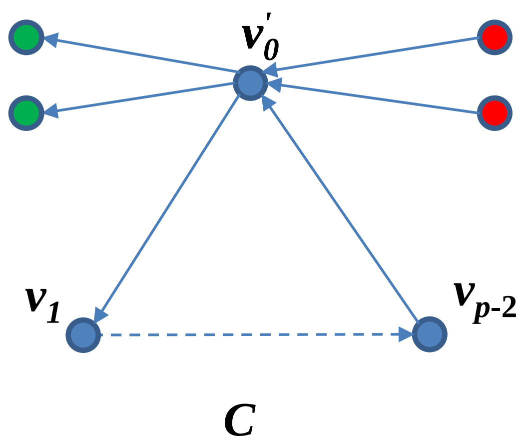
- **Size (Zhang et al., SIDMA 2017)**

  **If $D$ has at least $(n - 1)^2+1$ arcs, then $D$ is path extendable. with some exceptions.**

# Techniques

- **Counting the number of arcs between two parts of the digraph, or counting the degree sums.**
- **A contraction that builds up a corresponding between non-extendable paths and non-extendable cycles.**




**$P$ is a non-extendable path in the original digraph, if and only if $C$ is a non-extendable cycle in the resulted digraph.**

# Further Problems

- **Degree sum conditions for cycle extendability in digraphs.**
- **Degree sum conditions for path extendability in digraphs.**

# 05 Tournaments and Generalizations

# Special Classes of Digraphs

- **Path-mergeable digraphs.**
- **Locally in-semicomplete digraphs.**
- **Oriented graphs: digraphs without $2$-cycle.**

# Tournaments & Generalizations

- **Tournament:** **a digraph with EXACTLY one arc between every two distinct vertices.**
- **Bipartite tournament (BT):** **a $2$-partite tournament.**
- **Multipartite tournament (MT, or $k$-partite tournament):** **a digraph whose vertices can be partitioned into $k$ parts, with no arc between vertices in the same part, but exactly one arc between every two vertices in different parts.**
- **Semicomplete multipartite digraphs.**

# Tournaments 1

- **(Rédei, "Ein kombinatorischer Satz", 1934)**

  **Every tournament is traceable.**

- **(Camion, 1959)**

  **A strong tournament is Hamiltonian.**

- **(Moon, 1968)**

  **Every strong tournament is vertex-pancyclic.**

- **(Moon, 1969 & Hendry, 1989)**

  **A strong tournament is cycle extendable, unless it belongs to one exceptional class of tournaments.**

# Tournaments 2

- **(Hendry, 1989)**

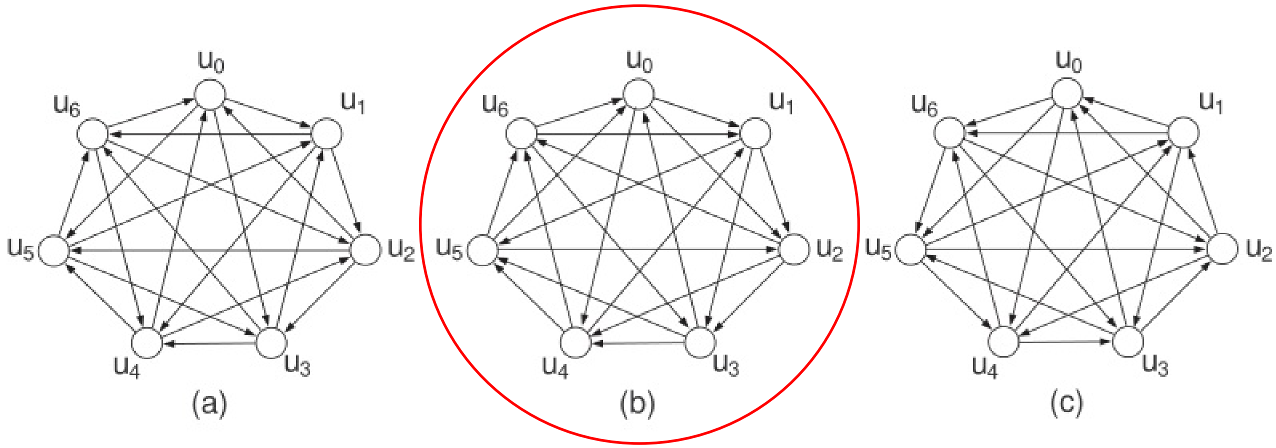
  **A regular tournament is cycle extendable, unless it belongs to one exceptional class of tournaments.**

- **(Zhang et al., SIDMA 2017)**

  **Tournaments are not generally path extendable.**

  **Neither are strong and regular tournaments.**

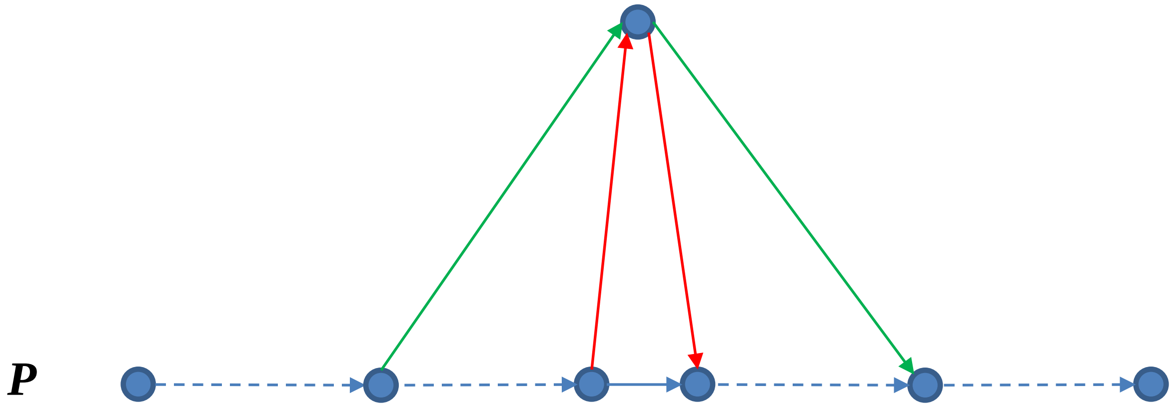
  **In a regular tournament every path of length at least 2 is extendable, unless it belongs to two exceptional class of tournaments, or isomorphic to a regular tournament on 7 vertices.**




The three regular tournaments
on 7 vertices, among which
only (b) is path extendable

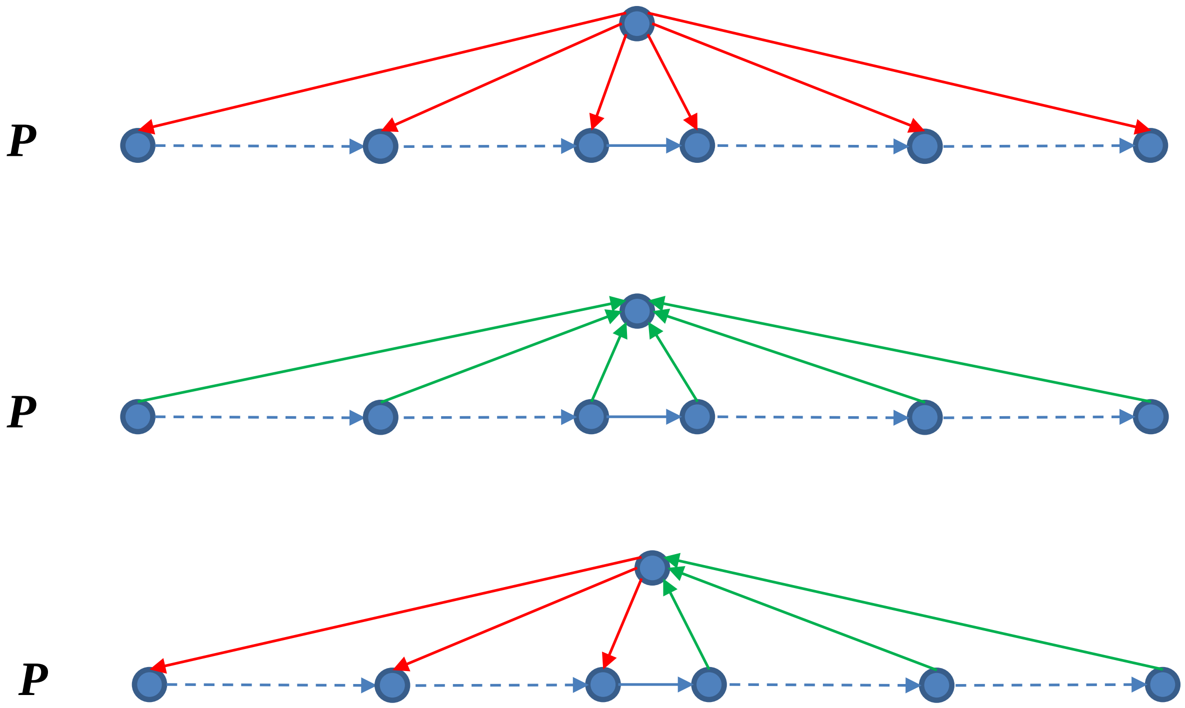
# Techniques

- **An impossible configure.**



# Techniques

- **Dominating, dominated and hybrid vertices.**



# Further Problems

- Generalize some existing conditions for pan-connected in tournaments to those for path extendability in tournaments.
- Strong cycle and path properties, such as pancyclicity, cycle extendability, panconnectedness and path extendability of multipartite tournaments.

Thanks
for your attention!